A COUPLED INTEGRAL EQUATION SOLUTION FOR MICROSTRIP TRANSMISSION LINES

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Abstract

A theoretical analysis of microstrip transmission lines is presented which is based on a derivation of a set of coupled integral equations in terms of a free-space Green's function in the related inhomogeneous region.

Introduction

During the last decade, several numerical solutions for microstrip transmission lines have been developed in terms of a two-dimensional quasi-static model. These solutions of the static model were formulated in terms of (1) conformal mapping approaches [1], (2) variational procedures [2], and (3) integral equation formulations using a "dielectric" Green's function or an "image" Green's function [3-6].

Bryant and Weiss [3] developed an exact or "dielectric" Green's function for the inhomogeneous medium of the transmission line of Figure 1, which consists of a dielectric

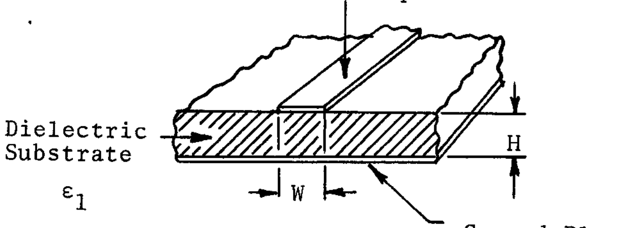

FIG. 1. Microstrip transmission line

substrate and a vacuum above. Their solution is based on the linear superposition of unique solutions of appropriate related problems to determine the desired solution. However, to obtain their "dielectric" Green's function, an inhomogeneous Fredholm integral equation of the second kind must be solved to determine an unknown charge distribution on the dielectric interface resulting from a unit strip source. The potential due to a unit source on the interface, i.e., the "dielectric" Green's function, is then computed from a derived integral relationship in terms of this charge distribution. An ordinary integral formulation of the composite problem is then solved using appropriate numerical procedures and the computed Green's function. The additional computations required to compute the Green's function results in a rather long computation time for this solution technique. Another approach to this problem used by Farrar and Adams [5], as well as others, which has a shorter computation time employs a technique whereby the Green's function is derived from the multiple images of a unit line source that satisfy the boundary conditions on the ground plane and at the dielectric interface. However, this "image" Green's function is limited to problems having simple geometries for which an image can be defined.

The purpose of this paper is to present a coupled integral equation formulation of the static microstrip transmission line in terms of a free-space Green's function which leads to a straight forward numerical solution by the method of momemts [7]. The approach is not limited to specific geometries, and arbitrary shaped, inhomogeneous TEM transmission lines can be analyzed with this boundary-value problem approach.

Integral Equation Formulation

A two-dimensional boundary-value problem is considered in this analysis which consists of two regions as shown in Figure 2.

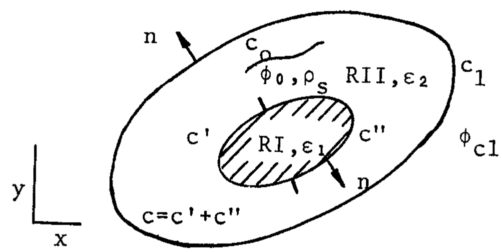

FIG. 2. Generalized static problem

Region RI consists of a region of homogeneous, isotropic dielectric material of permittivity, ε_1, bounded by the contour c where c=c'+c". Region RII consists of a homogeneous, isotropic region of permittivity, ε_2, bounded by the contours c and c_1 including a surface charge source ρ_s on contour c_0. In this case, the potential everywhere on c_0 and c_1 are assumed to be known and constant, i.e. ϕ_0 and ϕ_{c1}, respectively; however, neither ϕ nor $\partial\phi/\partial n$ are known on c.

If a general homogeneous region is considered, Laplace's equation,

$$\nabla^2\phi(x,y) = 0 \tag{1}$$

holds since the region is assumed to be source free, i.e. a boundary-value problem, and the free-space Green's function relation,

$$\nabla^2 G_0(x,y|x',y') = -\delta(x'-x,y'-y), \tag{2}$$

can be considered for each region where

$$G_0(x,y|x',y') = -\frac{1}{2\pi}\ \ln\sqrt{(x-x')^2 + (y-y')^2}. \tag{3}$$

It is well known that an integral equation for $\phi(x,y)$ can be derived by multiplying (1) by G_0 and (2) by ϕ, and then these relations are summed and Green's theorem is applied to obtain the relation

$$\phi(x,y) = \int_{c_T} (G_0 \frac{\partial \phi}{\partial n} - \phi \frac{\partial G_0}{\partial n}) ds \qquad (4)$$

where c_T is the total contour bounding the region. This result is an integral equation for the solution of Laplace's equation in terms of the potential and its normal derivative only on the bounding contours [8].

Thus, in the region RI, an integral equation of the form of (4) can be formulated using a free-space Green's function which is a function of the potential and the normal derivative of the potential. Note that the specification of both ϕ and $\partial\phi/\partial n$ (Cauchy boundary conditions) is an over specification of the problem.

A related equation for region RII can also be derived that is identical to (4) except the source ρ_s is included which is simply an integral equation for Poisson's equation [8].

In the class of problems solved here, it is assumed that the potential is known on all contours except on c and that the normal component of displacement must be continuous on c. If these boundary conditions are enforced between these two formulations on the appropriate contours c' and c", then the problem can be readily solved from the limiting case of these integrals through the use of potential theory. In effect, the two formulations are employed in a manner such that the potential on the contour c is eliminated from the integral equation formulations, and the normal derivative of the potential is expressed in terms of equivalent surface charge density on the contour c.

From (4), the potential equation for region RI results in two expressions as

$$\phi_1(x,y) = \int_{c} (G_0 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial G_0}{\partial n}) ds, (x,y) \varepsilon RI \qquad (5)$$

$$0 = \int_{c} (G_0 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial G_0}{\partial n}) ds, (x,y) \varepsilon RII, \qquad (6)$$

and from an integral equation solution of Poisson's equation [8] for Region RII

$$\phi_2(x,y) = \frac{1}{\varepsilon_2} \int_{c_0} \rho_s G_0 ds - \int_{c+c_1} (G_0 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial G_0}{\partial n}) ds, \quad (x,y) \varepsilon RII, \qquad (7)$$

$$0 = \frac{1}{\varepsilon_2} \int_{c_0} \rho_s G_0 ds - \int_{c+c_1} (G_0 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial G_0}{\partial n}) ds, \quad (x,y) \varepsilon RI \qquad (8)$$

where if (x,y) does not belong to the appropriate region, then the potential vanishes because the region of integration does not contain the delta function of (2). For the class of problems considered here, the contour c_1 is assumed to approach infinity, and, therefore, the integral on c_1 does not contribute to the solution.

The potentials ϕ_1 and ϕ_2 can be eliminated from the integrands of the potential integral equations (5-8) by using the boundary condition, $\phi_1 \equiv \phi_2$ on c in (6) and (8). If these four equations are combined, an equation for the potential in each of the two regions are obtained as follows:

$$\phi_1(x,y) = \frac{1}{\varepsilon_2} \int_{c_0} \rho_s G_0 ds + \int_{c} G_0 [\frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n}] ds, \quad (x,y) \varepsilon RI \qquad (9)$$

and

$$\phi_2(x,y) = \frac{1}{\varepsilon_2} \int_{c_0} \rho_s G_0 ds + \int_{c} G_0 [\frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n}] ds, \quad (x,y) \varepsilon RII, (10)$$

or $\phi_1 = \phi_2 = \phi$ everywhere in RI and RII. Since the integrand of the contour integral is discontinuous for (x,y) on c, these potential functions can be represented in terms of an equivalent surface charge density, σ. Thus, the two dielectric problems can be represented in terms of a source, ρ_s, and a surface charge density residing on the interface in a homogeneous medium ε_2 such that the potential everywhere is

$$\phi(x,y) = \frac{1}{\varepsilon_0} \int_{c_0} \rho_s G_0 ds + \frac{1}{\varepsilon_0} \int_{c} \sigma G_0 ds \qquad (11)$$

where $\varepsilon_2 = \varepsilon_0$.

If the assumed mathematical model consisting of the homogeneous region with sources ρ_s and σ is to completely represent the two dielectric problems, the normal component of displacement, D, must be continuous on c. Hence, another integral equation in terms of the equivalent surface charge density can be obtained from this continuity requirement from the derivatives of (11) in the appropriate region as

$$0 = (ke-1) \int_{c_0} \rho_s \frac{\partial G_0}{\partial n} ds + (ke+1) \frac{\sigma}{2}$$

$$+ (ke-1) \int\!\!\!\!-_{c} \sigma \frac{\partial G_0}{\partial n} ds, (x,y) \varepsilon c \qquad (12)$$

where the limiting case, $(x,y)\varepsilon c$, is assumed [9] and $ke = \varepsilon_1/\varepsilon_0$.

To solve the microstrip problem, the contour c_0 is moved to the surface c and made to correspond to c', for example; then the entire boundary c is distorted to have the shape of the microstrip and its image as shown in Figure 3. Equations (11) and (12) form a coupled set of integral equations

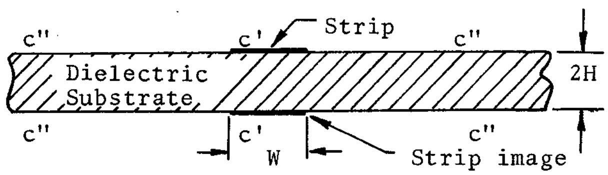

FIG. 3.Equivalent Microstrip Problem

which can be used to solve for the unknown charge density as c_0 approaches c'. In the limit as the source ρ_s is placed on the interface, the requirement that the normal component of displacement be discontinuous on the strip yields

$$0 = (ke-1) \frac{\rho_s}{2} + (ke+1) \frac{\sigma}{2}$$

$$+ (ke-1) \left[\int\!\!\!\!-_{c'} \rho_s \frac{\partial G_0}{\partial n} ds + \int\!\!\!\!-_{c} \sigma \frac{\partial G_0}{\partial n} \mathbf{ds} \right] , (x,y) \varepsilon c' \qquad (13)$$

where the limit as $c_0 \to c'$ is applied in the computation of D in each region. Thus, (11) and (13) are valid on the strip region c' and (12) holds for continuity of displacement on c''.

Results

The set of coupled integral equations of (11), (12), and (13) can be solved for the unknown free charge on the strip and for the equivalent bound charge on the dielectric interface using a method of moments approach [7]. Numerical results of a parameter study for this formulation obtained for a pulse function expansion of the unknown charge densities with point matching is presented in Table I for ke=16. The charge density distribution on the strip and the "bound" charge density distribution on the interface are shown in Figure 4 for W/H=1.0 with ke=4.0. The data from Table I show that this free-space Green's function solution is in close agreement with the results of other techniques which supports the validity of this integral equation formulation.

In conclusion, it is well to note that this type solution is extremely useful for the analysis of general inhomogeneous transmission lines of arbitrary cross section.

TABLE I
Characteristic Impedance of Microstrip Transmission Lines (ke=16)

W/H	Z_0	Z_0[3]	Z_0[5]
0.1	87.3058 Ω	87.680 Ω	86.9659 Ω
0.4	57.9962	57.841	57.4999
1.0	39.6272	39.272	39.2512
2.0	26.8795	26.644	26.7555
4.0	16.7348	-	16.7210
10.0	7.9992	-	8.0385

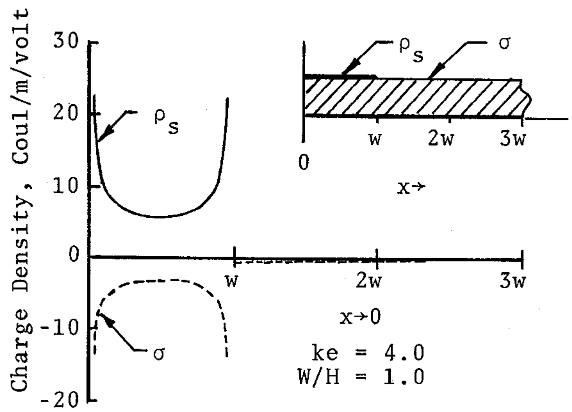

FIG. 4.Strip and "bound" charge density distribution; W/H=1.0 and ke=4.0.

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